# What is algebraic geometry?

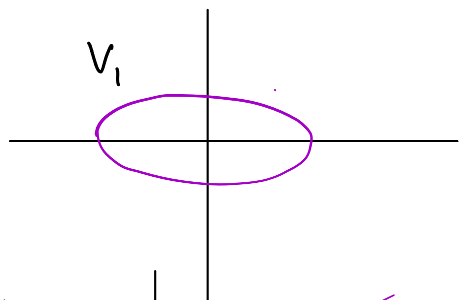
**Non-rigorous answer:** A "variety" is (locally) defined by polynomial equations in $k^n$, $k$ a field

$$V = \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0\} \subseteq k^n$$

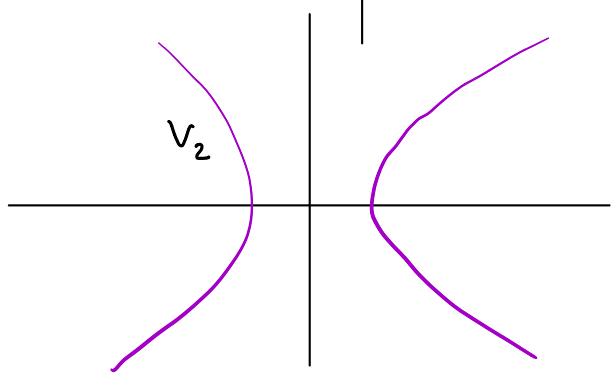
where the $f_i$ are polynomials in $k[x_1, \dots, x_n]$

**Ex:** Conics in $\mathbb{R}^2$, e.g.:

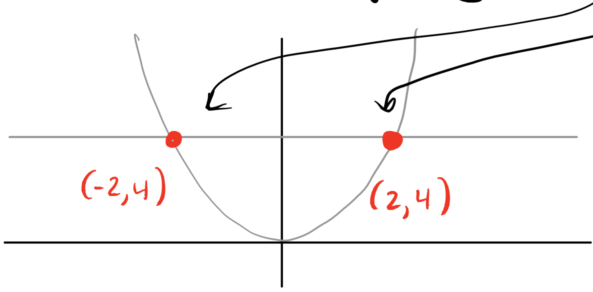
$V_1 = \{(x,y) \mid x^2 + 4y^2 - 1 = 0\}$
(ellipse)




$V_2 = \{(x,y) \mid x^2 - y^2 - 1 = 0\}$
(hyperbola)




**Ex:** $V_3 = \{(x,y) \mid y - x^2 = 0,\ y = 4\}$ = two points




For varieties in $k^n$, we will see that there is a "dictionary" between algebraic geometry and commutative algebra.

Not all varieties live in $k^n$, but we can often reduce AG questions to algebra questions by looking locally.
(simalar to real analysis vs. real manifolds)

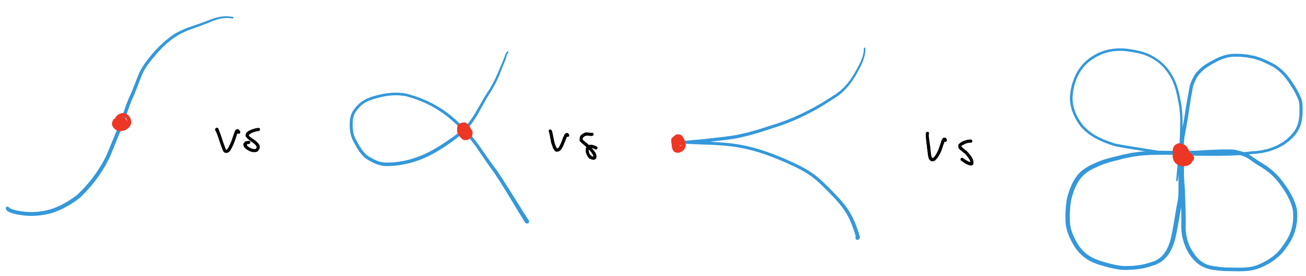
## What kinds of questions can we ask about varieties?

Let $V$ be a variety.

### a.) Singularity Theory

- What kind of geomety does $V$ have at a point?

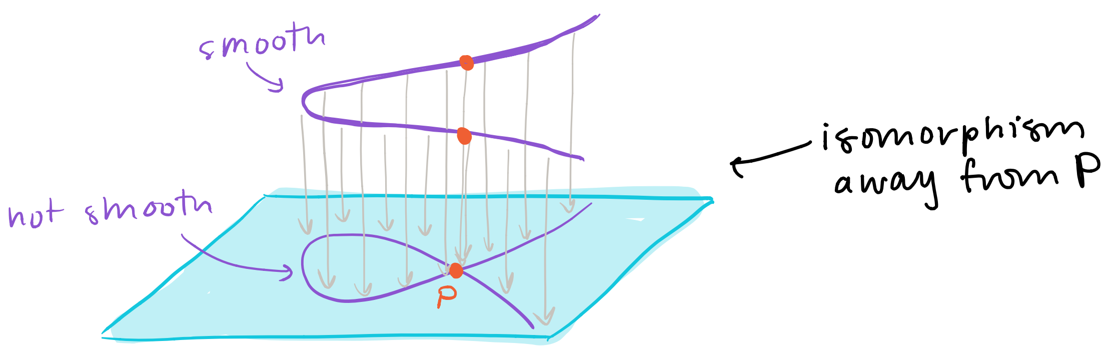
e.g.




All of these points appear to be different, but how is this reflected in the algebra?

- How to find a "smooth model" of a variety?

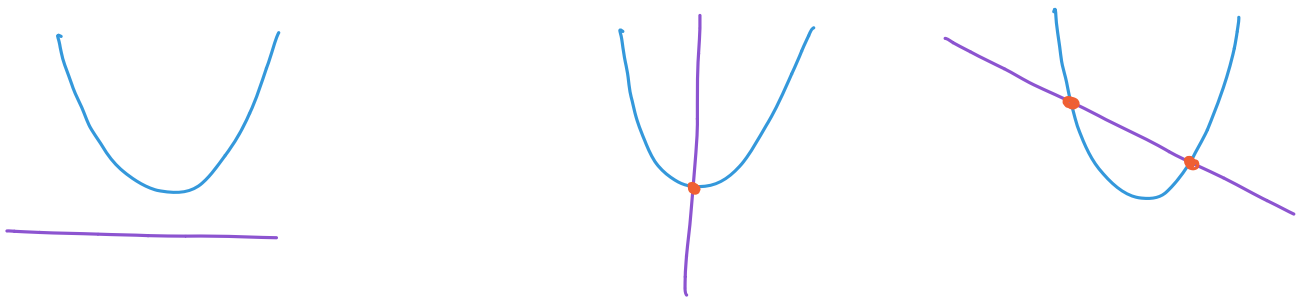
e.g.



## b.) Intersection theory

How do two (or 3, 4, ...) varieties intersect?

e.g. in the plane, a line and a conic can intersect in 0, 1, or 2 points:



We will see that in "nice" situations, a line and a conic always intersect in exactly 2 points, counting multiplicity (Bézout's Theorem).

## c.) Number Theory

counting rational points, e.g. Diophantine problems:

What are the rational solutions to $x^n + y^n = 1$?

Geometric translation: what does the corresponding variety in $\mathbb{Q}^2$ look like?

## d.) Embedding questions

Given a variety $V$, is there an embedding $V \hookrightarrow k^n$?
What about $V \hookrightarrow \mathbb{P}^n$?

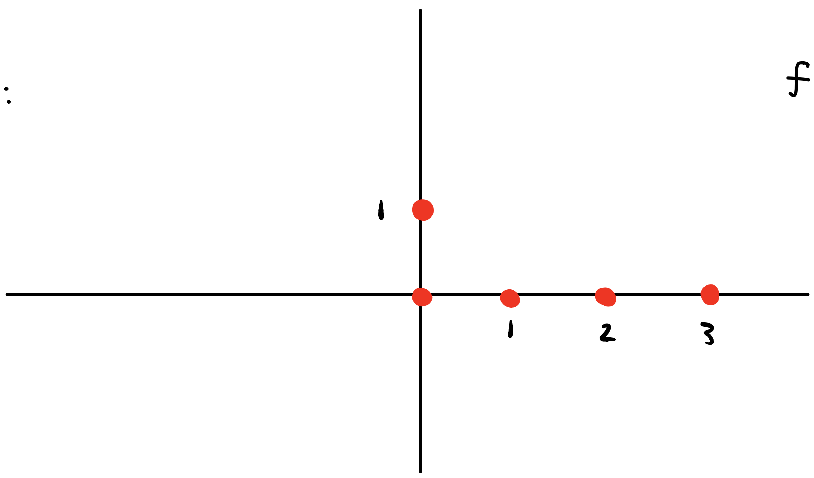
If so, what's the smallest $n$ s.t. an embedding exists?

## e.) Points imposing conditions on polynomials

e.g. if $P_1, \dots, P_6 \in \mathbb{R}^2$, which conics contain them?

Classical fact: Most sets of 5 points are contained in a unique conic. However, special ones are contained in more.

e.g. :




$f(x,y) = xy$ vanishes at all 5 points, but so does $g(x,y) = y(y-1)$.

In fact $y(y - ax - 1)$ will vanish at all 5 points for any $a \in \mathbb{R}$.